Formulating Equations for Word Problems

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In this paper we describe the ways in which students in Years 9 and 10 perceived problem structures and tried to represent them algebraically. We show that, for some students, different verbal descriptions of the same problem influenced the structures perceived and the non-algebraic solution strategies used. However students who used algebra were not affected by the form of the verbal description. They were able to transform their initial model to the one appropriate for algebra.

If students are to derive any power from the algebra they learn in school, they must be able to take a problem situation, recognize a mathematical structure in it, and formulate useful expressions and equations from it. As teachers know, many students have great difficulty in formulating algebraic equations to represent problem situations. A survey of textbooks indicates that students are given very little help in linking a problem with its algebraic formulation. The instructions for writing equations shown in Figure 1 illustrate how the difficulty of making the link is overlooked.

The first step in solving a problem is to put a pronumeral in place of the unknown number. The next step is to use the information in the problem to form an equation. Solve the equation to find the value of the unknown number.

Mathematics for Australian Schools Year 7, 1988, p. 338

1. Read the question carefully, twice.
2. Decide what you have to find.
3. Choose a letter to represent this unknown.
4. Draw a diagram if it will help.
5. Form an equation and solve it.

Mathematics Today Year 8, 1988, p. 196

First, represent what we have to find as a pronumeral. Second, write an equation which includes all the information stated in the problem. Third, solve the equation.

Nelson Maths 8, 1993, p. 379

(a) Let the unknown quantity be represented by a pronumeral.

(b) Write an equation.

Maths 9, (2nd edn.), 1993, p. 138

Use two pronumerals to stand for the two unknowns, create an algebraic model, and solve to find the values of the unknowns.

Mathematics Today Year 10 (2nd edn.), 1994, p. 77

Figure 1. Instructions for writing equations

None of the advice in these books gives the student any clues about how to 'create an algebraic model' or 'write an equation'. Beginners are frequently taught to write an algebraic equation by direct translation of key words in a statement of English. This direct translation procedure is only useful for a very limited range of simple problems. No general advice is available to help students formulate an equation in other contexts, where they may need to select, reorganize or transform information and make inferences.

Research on solution methods for algebra word problems has found that in some circumstances students use schema-driven approaches which direct them to fill 'slots' in a schema or 'frame' and then use a solution procedure known to apply to problems of that category. For problems that do not fall into a known category they may use a general solving procedure that starts with given information and operates deductively line by line. This procedure, which appears to be not intuitive and has to be taught, typically involves writing chains of logically equivalent statements. Another approach is to 'read' information from a mental model of the problem situation. In this approach, quantities and the relationships between them are represented in a non-verbal cognitive system, and logical reasoning is carried out by separating, moving and recombining parts of this representation.

There is considerable psycholinguistic evidence that comprehension of the relationships described in a problem involves constructing a mental model of some kind, and that mental operations may be carried out on this model. Paige and Simon (1966) noticed that students solving problems often referred to a mathematical object by different names (e.g., 'the total', 'the final', 'the mixture') and used names for a variable that referred to different levels of abstraction (e.g., 'value of the coins', 'number of coins', 'money', 'coins'). They suggest that these imprecise or inconsistent names came from a mental model from which assumptions and relationships were 'read off' directly. As the work of Johnson-Laird (1983) and MacGregor and Stacey (1993) has shown, such mental models represent perceived elements of a situation and salient relationships between them, and support reasoning. In this paper we describe students' mental models of problem structure and the extent to which their initial perceptions affected algebraic representation.

Aim of the Study

Alternative verbal descriptions of the same problem situations were used to prompt students to form different mental models. One set of mental models (a sum of parts) is compatible with algebraic solutions of the problems; the other (division into parts) is not. The study set out to explore whether:

(a) different verbal presentations of the same situation would lead students to construct different mental models;

(b) students recognized alternative presentations as relating to the same mental model;

(c) students attempted to express both models algebraically;

(d) students could 'move' from one model to the other;

(e) forming a mental model that was unhelpful for algebra was a cause of algebraic errors.

Test Items and Testing Procedure

The items shown in Figure 2 were included in two forms A and B of a pencil-and-paper test. The three items each describe problem situations where the size of a part is to be found, given information about the whole and comparisons between the parts. In Test A, each problem is described as a sum of parts. We expected that students would construct from this description a mental model reflecting its sum-of-parts structure, and solve it by a *subtract and divide* method. The sum-of-parts model is totally compatible with an algebraic formulation, such as $x + (x + 5) = 47$ for Item 1, and the *subtract and divide* method matches the algebraic solution of taking 5 from both sides and then dividing by 2. It also involves easy arithmetic. In Test B, Items 1 and 2 are described as a division into parts. We expected that students would construct a different, but equally correct, mental model and consequently tend to solve the problems by a strategy of *share equally, then adjust* . For Item 1B, for example, they would first allocate Mark and Jan equal shares ($23.50 each) and then try to adjust the shares by giving some of Jan's money to Mark. This method of solution is not compatible with a solution using algebraic equations. The arithmetic necessary is harder than for the sum-of-parts model, and for Item 2B after calculating $80 \div 3$ it is very difficult to see how to adjust the three distances. It was predicted that students working from a division-into-parts model would find the problems harder than those working from a sum-of-parts model and would be less successful in writing equations.

TEST A
1A. Jan has $x. Mark has $5 more than Jan has. Altogether they have $47. How much has each person got?
2A. A group of scouts did a 3-day walk on a long weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. The total journey was 80 km. How far did they walk on Saturday?
3. Jeff has to wash 3 cars. The second car takes 7 minutes longer than the first one, and the third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?
TEST B
1B. $47 is shared between two people, Mark and Jan. Jan gets $x. Mark gets $5 more than Jan gets. How much does each person get?
2B. A group of scouts walked a distance of 80 km on a 3-day weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. How far did they walk on Saturday?
3. Jeff has to wash 3 cars. The second car takes 7 minutes longer than the first one, and the third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?

Figure 2. The two versions of the test

The test was given in seven mixed-ability classes in two schools to 166 students in Years 9 and 10. Test papers were randomly distributed, half the students receiving Test A and half receiving Test B. Test B (expected to be the harder version) was also given to a selected class of 28 high achievers in Year 9 in another school. At the beginning of each test paper there were instructions to write an equation for each item and solve it.

Results and Discussion

When interpreting students' responses, we assumed that (i) a student's written work is an indication of the mental model constructed and the reasoning processes used to work out the answer; and (ii) if the written work suggests the substitution of one model by another (e.g., by symbols being crossed out), then the first expression written indicates the problem structure that was perceived first.

In the class of high achievers there was only one error over all items (the error was in Item 2B). Data from this class has not been included in Tables 1 and 2. Table 1 shows the percentages of correct answers to each problem (regardless of method used) for the mixed-ability sample. It may be seen that students doing Test A were more successful than students doing Test B, in accordance with our expectations

Table 1 Success rates on alternative versions of three test items (N = 166)

Version	*n*	Item 1	Item 2	Item 3
A	83	73%	73%	73%
B	83	67%	64%	63%

Table 2 also refers only to the mixed-ability sample. It shows the number of students carrying out subtraction or division as their first written operation on each item, and whether or not their final answer was correct. Some students began an item in one way (e.g., dividing 80 by 3 in Item 2B) and then changed strategy (e.g., to a sum-of-parts subtraction or a guess-and-check method) to obtain their final answer. Very few students wrote equations and solved them. When they did so, their solution methods have been classified under 'Subtract' in Table 2 since a *subtract and divide* method is implied. Solution methods used by the high-achieving class are shown in Table 3.

Table 2 Numbers of correct and incorrect solutions related to first operation written down and version of test (mixed-ability classes, N = 166)

	Test A						Test B					
	Subtract		Divide		Other		Subtract		Divide		Other	
Item	Corr	Inc	Corr	Inc	Corr	Inc	Corr	Inc	Corr	Inc	Corr	Inc
1	34	3	18	15	9	4	22	0	30	27	4	0
2	42	8	3	3	16	11	37	9	5	14	11	7
3	44	6	3	3	14	13	38	5	2	11	12	15

[a]Category 'Other' includes answer only, guess-and-check method, method not clear, and no answer.

Table 3 Solution methods for Test B items (high achievers, N = 28)

	Algebra	Subtract	Divide	Guess&Check	Not shown
Item 1	14	6	4	2	2
Item 2	22	4	0	1	1
Item 3	24	2	0	2	0

Association between Problem Presentation and Students' Success

As shown in Table 1, in the mixed ability sample there were more correct answers for Test A than for Test B. Combining data in Table 1 for all items, the chi-square test shows an association significant at the 5% level between the test version and success on these items [$chi^2(1, N = 498) = 4.5, p = 0.03$]. Since the two groups (A and B) were well matched, it is reasonable to conclude that the difference in difficulty was due to the different presentations. For the sample of high achievers, however, both versions of the problems were equally easy; there was only one error and that was caused by a calculation mistake during manipulation of an equation.

Association between Problem Presentation and Perceived Structure

Table 2 links the problem presentations to the mental models constructed, and refers only to the mixed-ability sample. It shows that 89 responses to Test B indicated division as the first operation whereas only 45 responses to Test A did so. Similarly, subtraction was indicated more often for Test A (137) than for Test B (111). The association between the test version and the first operation chosen is highly significant [$chi^2(1, N = 382) = 15.5, p = 0.0001$]. This finding supports the hypothesis that problem presentation is a factor influencing which mental model is constructed. This was not the case for the high achievers. As Table 3 shows, six high achievers who chose an arithmetic method for Item 1 used the *subtract and divide* method, despite the fact that the problem was presented as a division into parts and therefore expected to prompt the *share equally then adjust* strategy. For Item 2, also presented as a division into parts, almost all the students in this class wrote an equation and solved it, indicating that they were able to access a sum-of-parts model.

Effect of Mental Set

Since Item 3 is identical in both tests, the discrepancy between success rates on this item (see Table 1) is likely to be caused by a mental set favouring the model induced by the two previous items. The written working for Test B shows that most students had used the same procedure for Item 3 that they had used for Item 2, often written in exactly the same format. They perceived Items 1, 2 and 3 as having identical structures, although the first two were presented as a division into parts and the third was presented as a sum of parts.

Success in Writing Equations

Despite the explicit instruction to write an equation for each item, most students in the mixed-ability sample made no attempt to use algebra. The small number of equations written (14% of the possible total) makes it unwise to draw firm conclusions about whether students doing Test A (which prompted the model compatible with algebraic solutions) found it easier to construct equations than students doing Test B. However, there appears to be no difference. Some students wrote useful expressions for the three parts but did not know how to relate them to the total (e.g., Fig. 3 examples (ii) and (iii)). Others used three variables and did not know how to proceed (e.g., Fig. 3, example (vi)). The form of most responses suggests that the students who were prepared to try to write equations were able to access mental models incorporating the sum of parts, as is required for an algebraic solution. There was only one instance of an attempt to express algebraically a *share equally, then adjust* strategy based on the division into parts model (Fig. 3, example (v)).

Knowledge of algebra, or willingness to use it, varied considerably between classes and schools. We have seen that in the class of high achievers almost all students used algebra and knew how to use it correctly. In one of the two other schools, most students avoided any attempt to use it. In the other school, equations or algebraic expressions were written (but not necessarily used) in approximately 25% of responses. Several of these students used algebraic letters to represent elements of the problem (as shown in Fig. 3) but did not know how to combine them to form an equation. After finding that their algebra was not helpful, they switched to a guess-and-check procedure or arithmetic reasoning. It appears that they see algebra as a language for labelling unknown quantities or recording information about a mathematical relationship, but do not realise that it is also useful for solving problems.

i	$x + 7 + 13 = 80$	ii	x Sun, y Sat, z Mon $x = y + 7, z = y + 13, y =$Total $= 80$
iii	Sun $x + 7$ Mon $x + 13$ Sat x	iv	$x \text{->} x + 7 \text{->} x + 13 = 80$
v	$x \div 3 + 7 =$ $x \div 3 =$ $x \div 3 + 13 =$	vi	$y - 7 = x$ $z - 13 = x$ $(y - 7)(z - 13) = 2x$

Figure 3. Students' use of algebraic letters to record information for Item 2

Fluid Mental Representations

The results support the hypothesis that the different problem presentations promoted the construction of different mental models. However, as Table 2 shows, a considerable proportion of Test A students appear to have initially perceived the problems as division into parts and many Test B students initially perceived them as a sum of parts. Moreover, students who used a *share equally then adjust* strategy for Item 2 (adjusting the parts by guess-and-check) recognized that this procedure was relevant for Item 3 despite its presentation as a sum of parts.

It appears that attempting to use algebra forced students to change their initial perception of problem structure when it did not support an algebra formulation. The algebraic solutions written by the high achievers, which for the first two items did not conform with the problem presentation, strongly suggest that the learned routine of setting up an algebraic solution to a problem may guide students' perception and their construction of a mental model.

Implications for Teaching

We have shown that the structure of a problem can be perceived in different ways, depending on the form of its presentation and the method to be used for solving it. Within even simple problems such as the ones presented in this paper, there is a complex web of relationships between quantities which different students will perceive with different emphases and interpretations. Some perceived structures lead to easy solutions and others to difficult solutions. It is important for teachers to be aware of the variety of mental models their students may construct and to appreciate that routine procedures (such as solving problems algebraically) are compatible with only some of these models. Students need to know that there are alternative models of a situation, and that their initial perceptions of underlying structure may not be the most useful. Learned routines for setting up an algebraic equation guide students to select a model that is compatible with an algebraic solution.

References

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